Expansion of Signal Space to Include External Noises

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Introduction

The author has established a signal space theory that is based on Tangent Square Summation (TSS) theorem [1]. The theorem tells how a number of independent signals can be combined to a representative vector. The thermal noise naturally added to the receive signals are also to be incorporated in the signal space.　The thermal noise has a distinct feature that it is uncorrelated with any other signal. Therefore it adds another dimension to the signal space. This memorandum describes how the thermal noise can be incorporated into the signal space.

The thermal noise expands the dimension of the signal space that helps to avoid the trivial zero output problem in the interferences cancellation systems based on Least Mean Square Error (LMSE) algorithm. The noise generally unfavorable in communication, functions in this case as a stabilizing factor in the interference cancellation receivers.

１． Signals and Signal Space

Suppose we have signals Sd, S1, S2, ,,, Sm from different sources. Then they form a signal space with each signal giving the bases of the space. .

Any signal in the communication system is a combination of those signals originating from different sources.

Suppose

X = x1 ・ S1 + x2 ・ S2 + ,,, xm ・ Sm

Y = y1 ・ S1 + y2 ・ S2 + ,,, ym ・ Sm

Then the correlation or the inner product of the signals is;

(X,Y) = x1 ・ y1* + x2 ・ y2* +,,,,, xm ・ ym*

Thus the signals X and Y can be expressed as vectors in the signal space;

X = (x1, x2, x3, ,,,,, xm)

Y = (y1, y2, y3 ,,,,, ym)

2. Interference cancellation system

Suppose we have a desired signal Sd to receive and regenerate for communication. There are also other signals S1,S2,,,,Sm generated by different sources that leak into the receive circuit. In order to cancel those interferences, we set a number of auxiliary receivers. Let us denote the main receiver by X that is to receive the desired signal Sd, and the auxiliary receivers Y1,Y2,,,Yn to receive the interferences signals.

The main path and auxiliary paths signals are combinations of those signals;

X = Sd + I1・S1 + I2 ・ S2 +,,,,+ Im ・ Sm

Yi = Di ・ Sd + Li1 ・ S1+Li2 ・ S2 +,,,, + Lim ・ Sm (i = 1,2,,,,,n)

where Sd and {Sj ; j = 1,2,,,m} are original signals. Without loss of generality we assume the norms of original signals are normalized: ||S||=1.

The {Ii, Lij: i=1,2,,,n, j=1,2,,,,m} are transmit coefficients of the communication paths.

Least Mean Square Error Method (LMSE)

In order to cancel the interference signals, we subtract a combination of the auxiliary paths signals with adaptive weights to get the compensated signal Z.

Z = X – [i=1,n]∑Wi ・ Yi

where {Wi; I = 1,2,,,,n} are the adaptive weights.

We assume the power of signal Z will be **minimal** when the intended interferences cancellation is achieved. We control the adaptive weights Wi (I = 1,2,,,,n) to minimize ||Z||^2.

To do that we set the partial derivatives of ||Z||^2 by Wi*.

∂||Z||^2 /∂Wi* = 0

Then we get;

(Z, Yi) = 0 (i = 1,2,,,,n)

That is, the output signal must be orthogonal to all the auxiliary paths signals.

The weights {Wi} can be derived from the equation.

[k=1,n]∑(Yk, Yi) ・ Wk = (X, Yi) (i = 1,2,,,,n)

The equations can be expressed more simply;

[(Yk,Yi)] ・ [Wk> = [(X,Yi)> (k, i = 1,2,,,, n)

where [(Yk,Yi)] is an n x n matrix with (Yk,Yi) as its (i,k) elements and [Wk> a column (vertical) vector with Wk as the k-th element. Note the [(Yk,Yi)] is an Hermite matrix; [(Yk,Yi)] = [(Yi,Yk)]*

3. Tangent Square Summation Theorem

As the output Z must be orthogonal to all auxiliary path signals {Yi}, it must be orthogonal to the subspace spanned by those auxiliary paths signals.

$$Yi = Di \cdot Sd + Li1 \cdot S1 + Li2 \cdot S2 +,,,, + Lim \cdot Sm \qquad (i = 1,2,,,,,n)$$

Case of ideal auxiliary paths receivers;

We first analyze an ideal case that the number of the auxiliary receivers is the same as the number of interferences signals; n = m, and each auxiliary path picks purely the targeted interference signal.

$$Yi = Di \cdot Sd + Si \qquad (i = 1,2,,,,,m)$$

The subspace spanned by the auxiliary path signals is their linear combination with coefficients {wi};

$$Y = [i=1,m]\sum wi \cdot Yi \quad = ([i=1,m]\sum wi \cdot Di).Sd \quad + \quad [i=1,m]\sum wi \cdot Si$$

The tangent square of Y is given by

$$\tan^2(\theta Y) = \left| [i]\sum wi \cdot Di \right|^2 / ([i]\sum \left| wi \right|^2) \qquad \text{(To be maximized by wi; i = 1,2,,,,m)}$$

Note $\| Sd \| = \| Si \| = 1$

By setting

$$\partial/\partial wi^* = 0 \qquad (i=1,2,,,,m)$$

We get

$$Di^* \cdot ([i]\sum \left| wi \right|^2) - \quad wi \cdot ([i]\sum wi \cdot Di) \quad = 0$$

Or

$$wi / Di^* = ([i]\sum \left| wi \right|^2) / ([i]\sum wi \cdot Di) \qquad (i = 1,2,,,,m)$$

They must be all equal to a common value, say K

$$wi / Di^* = K \qquad (i=1,2,,,,m)$$

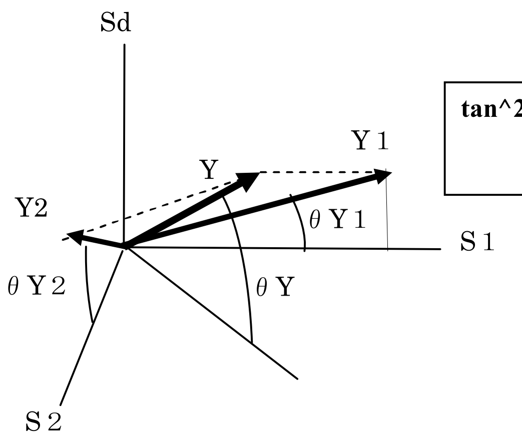
Which gives

$$\mathbf{\tan^2(\theta Y)} = [i=1,m]\sum \left| Di \right|^2 \quad \mathbf{=} \quad \mathbf{[i=1,m]\sum \tan^2(\theta Yi)}$$

Note

$$\tan^2(\theta Yi) = \| Di \cdot Sd \|^2 / \| Si \|^2 \quad = \left| Di \right|^2$$

A graphical depiction is given in the following figure for the simple case of two signals..



$\boxed{\begin{aligned} \mathbf{tan^{\wedge}2(\theta Y)} &= \mathbf{tan^{\wedge}2(\theta Y1)} \\ &\quad + tan^{\wedge}2(\theta Y2) \end{aligned}}$

Tangent Square Summation Theorem

Cases in general;

We have the following situation;

Yi = Di・Sd + Li1・S1+Li2・S2 +,,,, + Lim・Sm (i = 1,2,,,,,n)

In vector forms;

[Y> = [D>・Sd + [L][S>

Where [Y>, [D>, [S> are column vectors with the i-th elements are respectively Yi, Di, Si. And [L] is the matrix whose (i,m) component is Lim.

If [L] is **regular**, the above equation is applied with the inverse matrix [/L],

[Y'> = [/L]・[Y> = [D'>・Sd + [S>

Where

[/L]・[D> = [D'>

Or

[L]・[D'> = [D>

The above situation is now the same as the special case which tells;

tan^2(θY') = [i=1,m]∑ ｜Di'｜^2 **= [i=1,m]∑ tan^2(θYi')**

The above operations are linear combinations of the auxiliary paths vectors, which do not alter the structure of the signal subspace {Yi'} = {Yi}, hence

tan^2(θY) = tan^2(θY')

and the TSS theorem holds for the general cases.

4. Representative vector of the auxiliary subspace;

The auxiliary paths signals {Yi; i=1,2,,,,n} form vectors in the signal space {Sd, {Sj}}.

Two vectors Yi and Yj span a plane in the signal space by a linear combination;

α・Yi + β・Yj

where α, β are arbitrary complex numbers.

The spanned plane is **represented** by the vector

Y(i,j) =α(i,j)・Yi + β(i,j)・Yj

which maximizes the power ratio of the Sd component to that of the {Si} components[1].

The above process is repeated to get the vector Y(1,2,,,n) which represents the signal space formed by the auxiliary path signals.

Thus the subspace of the auxiliary paths receivers can be represented by a single vector as if being just a one-dimensional subspace.

5. Inclusion of Thermal noise into Signal Space

The main path signal X generally contains the desired signal Sd, the interference signals S1,S2,,,,Sm and the thermal noise Nx.

The auxiliary paths signals {Y1,Y2,,Yn} each contain thermal noise Ni mainly originated at the receiver antenna and the front head Low Noise Amplifiers (LNA).

The main path and auxiliary paths signals are now expressed as follows;

$X = Sd + I1 \cdot S1 + I2 \cdot S2 +,,,,+ Im \cdot Sm + Nx$

$Yi = Di \cdot Sd + Li1 \cdot S1 + Li2 \cdot S2 +,,,, + Lim \cdot Sm + Ni \qquad (i = 1,2,,,,,n)$

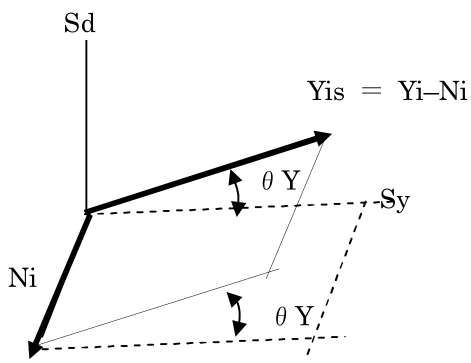
How do we incorporate the thermal noise Ni into the signal space?

This problem is readily solved by TSS theorem as follows. The signal Yi is decomposed into two components;

$Yi = (Yi - Ni) + Ni = Yis + Ni \qquad (Yis = Yi - Ni)$

The first term contains all the signals Sd, S1, S2,,,,Sm and the second term only the thermal noise. The thermal noise term contains no Sd, hence its tangent angle is zero. Then by TSS theorem the tangent angle of Yis is equal to that of Yi. In another words the additional thermal noises expand the dimensions of the signal space but do not add any tangent square value of the auxiliary paths signal space.

The above description is readily depicted in the following figure.



Inclusion of thermal and external noises

Problem;

The **representative vector** of the above subspace (plane) spanned by Yis (=Yi-Ni) and Ni is apparently Yis itself which does not include the thermal noise Ni.

How can we get the subspace with the representative vector Yi including the noise?

Solution;

Let us first review Yi-Ni without the external noise.

$Yis = Di \cdot Sd + Li1 \cdot S1 + Li2 \cdot S2 +,,,, + Lim \cdot Sm = Di \cdot Sd + Syi$

Where

$Syi = Li1 \cdot S1 + Li2 \cdot S2 +,,,, + Lim \cdot Sm$

Let us modify the above equation as follows;

$Yis / \| Syi \| = Di / \| Syi \| \cdot Sd + Syi / \| Syi \| = Dii \cdot Sd + Syi / \| Syi \|$

Where

$Dii = Di / \| Syi \|$

Then the tangent of Yis is

$$\tan^2(\theta Yis) = |Di|^2 / \| Syi \|^2 = |Dii|^2 = \tan^2(\theta Yi) \text{ (by TSS theorem)}$$

Let us now express Yi in the following form;

$$Yi = \alpha \cdot Yis' + \beta \cdot Yin'$$

$$= Dii \cdot (\alpha . \cos(\phi) + \beta . \sin(\phi)) \cdot Sd + \alpha . Syi / \| Syi \| + \beta . Ni / \| Ni \|$$

where

$$Yis' = Dii \cdot \cos(\phi) \cdot Sd + Syi / \| Syi \|$$

$$Yin' = Dii \cdot \sin(\phi) \cdot Sd + Ni / \| Ni \|$$

The coefficients α and β are determined to maximize the $\tan^2(\theta Yi)$

$$\tan^2(\theta Yi) = |Dii \cdot (\alpha . \cos(\phi) + \beta . \sin(\phi))|^2 / \{ |\alpha|^2 + |\beta|^2 \}$$

To be maximized by coefficients α and β.

By differentiating

$$\partial \{\tan^2(\theta Yi)\} / \partial \alpha* = 0$$

$$\partial \{\tan^2(\theta Yi)\} / \partial \beta* = 0$$

We get

$$\alpha = \cos(\phi)$$

$$\beta = \sin(\phi)$$

And

$$Yi = Dii \cdot Sd + \cos(\phi). Syi / \| Syi \| + \sin(\phi). Ni / \| Ni \|$$

It is obvious that

$$\tan^2(\theta Yi) = |Dii|^2 = \tan^2(\theta Yis)$$

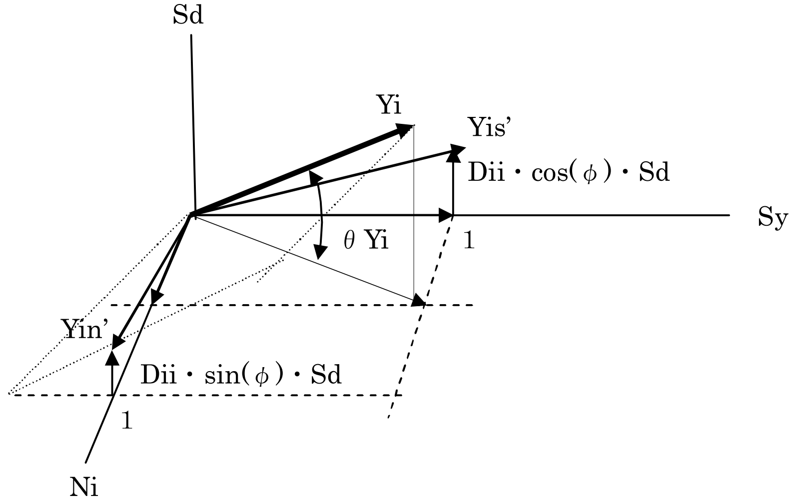
The $\cos(\phi)$ and $\sin(\phi) = \cos(\pi/2 - \phi)$ are directional cosines of the vector Yi against the normalized vectors $Syi / \| Syi \|$ and $Ni / \| Ni \|$ on the Syi x Ni plane. The case $\phi = 0$ corresponds to the situation of no thermal noise. The case $\phi = \pi/2$ corresponds to the situation Yi contains no Syi but only Ni and Sd.

The objective of the receiver Yi is to collect the interference signals Syi hence the signal-to-noise power ratio $(S/N)ii$ is

$$(S/N)ii = | \cos(\phi). Syi / \| Syi \| |^2 / | \sin(\phi). Ni / \| Ni \| |^2 = \cot^2(\phi)$$

Thus the inclusion of the thermal noise has been formulated as a rotation of the representative vector by an angle $\phi = \mathrm{arccot}(\sqrt{(S/N)ii})$ around Sd axis.

The expansion of the dimension of the signal space by inclusion of the thermal noise is depicted in the following figure.



Inclusion of thermal noises

Reference

[1] Osamu Ichiyoshi " A Signal Space Analysis of Interferences cancellation Systems" 2016 Joint Conference on Satellite Communications JCSAT 2016, IEICE Technical Report **SAT2016-60, pp147-152**