Introduction
The author has established a signal space theory that is based on Tangent Square Summation (TSS) theorem [1]. The theorem tells how a number of independent signals can be combined to a representative vector. The thermal noise naturally added to the receive signals are also to be incorporated in the signal space. The thermal noise has a distinct feature that it is uncorrelated with any other signal. Therefore it adds another dimension to the signal space. This memorandum describes how the thermal noise can be incorporated into the signal space. The thermal noise expands the dimension of the signal space that helps to avoid the trivial zero output problem in the interferences cancellation systems based on Least Mean Square Error (LMSE) algorithm. The noise generally unfavorable in communication, functions in this case as a stabilizing factor in the interference cancellation receivers.
1. **Signals and Signal Space**

Suppose we have signals $S_d, S_1, S_2, \ldots, S_m$ from different sources. Then they form a signal space with each signal giving the bases of the space.

Any signal in the communication system is a combination of those signals originating from different sources. Suppose

\begin{align*}
X &= x_1 \cdot S_1 + x_2 \cdot S_2 + \ldots + x_m \cdot S_m \\
Y &= y_1 \cdot S_1 + y_2 \cdot S_2 + \ldots + y_m \cdot S_m
\end{align*}

Then the correlation or the inner product of the signals is;

\[(X,Y) = x_1 \cdot y_1^* + x_2 \cdot y_2^* + \ldots + x_m \cdot y_m^*\]

Thus the signals $X$ and $Y$ can be expressed as vectors in the signal space;

\begin{align*}
X &= (x_1, x_2, x_3, \ldots, x_m) \\
Y &= (y_1, y_2, y_3, \ldots, y_m)
\end{align*}

2. **Interference cancellation system**

Suppose we have a desired signal $S_d$ to receive and regenerate for communication. There are also other signals $S_1, S_2, \ldots, S_m$ generated by different sources that leak into the receive circuit. In order to cancel those interferences, we set a number of auxiliary receivers. Let us denote the main receiver by $X$ that is to receive the desired signal $S_d$, and the auxiliary receivers $Y_1, Y_2, \ldots, Y_n$ to receive the interferences signals.

The main path and auxiliary paths signals are combinations of those signals;

\begin{align*}
X &= S_d + I_1 \cdot S_1 + I_2 \cdot S_2 + \ldots + I_m \cdot S_m \\
Y_i &= D_i \cdot S_d + L_{i1} \cdot S_1 + L_{i2} \cdot S_2 + \ldots + L_{im} \cdot S_m \quad (i = 1, 2, \ldots, n)
\end{align*}

where $S_d$ and $\{S_j ; j = 1, 2, \ldots, m\}$ are original signals. Without loss of generality we assume the norms of original signals are normalized: $||S||=1$.

The $\{I_i, L_{ij}; i=1, 2, \ldots, n, j=1, 2, \ldots, m\}$ are transmit coefficients of the communication paths.

**Least Mean Square Error Method (LMSE)**

In order to cancel the interference signals, we subtract a combination of the auxiliary paths signals with adaptive weights to get the compensated signal $Z$.

\[Z = X - \sum_{i=1}^{n} W_i \cdot Y_i\]

where $\{W_i; i=1, 2, \ldots, n\}$ are the adaptive weights.

We assume the power of signal $Z$ will be **minimal** when the intended interferences cancellation is achieved.

We control the adaptive weights $W_i (i = 1, 2, \ldots, n)$ to minimize $||Z||^2$.

To do that we set the partial derivatives of $||Z||^2$ by $W_i^*$.

\[\frac{\partial ||Z||^2}{\partial W_i^*} = 0\]

Then we get;

\[(Z, Y_i) = 0 \quad (i = 1, 2, \ldots, n)\]

That is, the output signal must be orthogonal to all the auxiliary paths signals.

The weights $\{W_i\}$ can be derived from the equation.

\[\sum_{k=1}^{n} (Y_k, Y_i) \cdot W_k = (X, Y_i) \quad (i = 1, 2, \ldots, n)\]

The equations can be expressed more simply;

\[\begin{bmatrix} (Y_k, Y_i) \end{bmatrix} \cdot [W_k]^* = [(X, Y_i)^*] \quad (k, i = 1, 2, \ldots, n)\]

where $[(Y_k, Y_i)]$ is an $n \times n$ matrix with $(Y_k, Y_i)$ as its $(i,k)$ elements and $[W_k]^*$ a column (vertical) vector with $W_k$ as the $k$-th element. Note the $[(Y_k, Y_i)]$ is an Hermite matrix; $[(Y_k, Y_i)] = [(Y_i, Y_k)]^*$
3. **Tangent Square Summation Theorem**

As the output $Z$ must be orthogonal to all auxiliary path signals $\{Y_i\}$, it must be orthogonal to the subspace spanned by those auxiliary paths signals.

$$Y_i = D_i \cdot S_d + L_1 \cdot S_1 + L_2 \cdot S_2 + \ldots + L_m \cdot S_m \quad (i = 1,2,\ldots,n)$$

**Case of ideal auxiliary paths receivers:**

We first analyze an ideal case that the number of the auxiliary receivers is the same as the number of interferences signals; $n = m$, and each auxiliary path picks purely the targeted interference signal.

$$Y_i = D_i \cdot S_d + S_i \quad (i = 1,2,\ldots,m)$$

The subspace spanned by the auxiliary path signals is their linear combination with coefficients $\{wi\}$;

$$Y = \sum_{i=1}^{m} w_i \cdot Y_i = (\sum_{i=1}^{m} w_i \cdot D_i) \cdot S_d + \sum_{i=1}^{m} w_i \cdot S_i$$

The tangent square of $Y$ is given by

$$\tan^2(\theta_Y) = \frac{\sum_{i=1}^{m} |w_i| \cdot |D_i|^2}{\sum_{i=1}^{m} |w_i|^2} \quad (To \ be \ maximized \ by \ w_i; \ i = 1,2,\ldots,m)$$

Note $\parallel S_d \parallel = \parallel S_i \parallel = 1$

By setting

$$\frac{\partial}{\partial w_i^*} = 0 \quad (i=1,2,\ldots,m)$$

We get

$$D_i^* \cdot (\sum |w_i|^2) - w_i \cdot (\sum w_i \cdot D_i) = 0$$

Or

$$w_i / D_i^* = (\sum |w_i|^2) / (\sum w_i \cdot D_i) \quad (i = 1,2,\ldots,m)$$

They must be all equal to a common value, say $K$

$$w_i / D_i^* = K \quad (i=1,2,\ldots,m)$$

Which gives

$$\sum_{i=1}^{m} |D_i|^2 = \sum_{i=1}^{m} \sum_{i=1}^{m} \tan^2(\theta_{Y_i})$$

Note

$$\tan^2(\theta_{Y_i}) = \frac{|D_i \cdot S_d|^2}{|S_i|^2} = \frac{|D_i|^2}{|S_i|^2}$$

A graphical depiction is given in the following figure for the simple case of two signals.
Cases in general:
We have the following situation;
\[ Y_i = D_i \cdot S_d + L_i \cdot S_1 + L_i2 \cdot S_2 + \ldots + L_i \cdot S_m \quad (i = 1, 2, \ldots, n) \]
In vector forms;
\[ [Y] = [D] \cdot [S] + [L] \cdot [S] \]
Where \([Y], [D], [S]\) are column vectors with the i-th elements are respectively \(Y_i, D_i, S_i\). And \([L]\) is the matrix whose \((i,m)\) component is \(L_{im}\).
If \([L]\) is regular, the above equation is applied with the inverse matrix \([/L]\),
\[ [Y'] = [/L] \cdot [Y] = [D'] \cdot S_d + [S] \]
Where
\[ [/L] \cdot [D] = [D'] \]
Or
\[ [L] \cdot [D'] = [D] \]
The above situation is now the same as the special case which tells;
\[ \tan^2(\theta Y') = \sum_{i=1}^{m} \left| D_i' \right|^2 = \sum_{i=1}^{m} \tan^2(\theta Y_i') \]
The above operations are linear combinations of the auxiliary paths vectors, which do not alter the structure of the signal subspace \(\{Y_i\} = \{Y_i\}\), hence
\[ \tan^2(\theta Y) = \tan^2(\theta Y') \]
and the TSS theorem holds for the general cases.

4. Representative vector of the auxiliary subspace;
The auxiliary paths signals \(\{Y_i; i=1,2,\ldots,n\}\) form vectors in the signal space \(\{S_d, \{S_j\}\}\).
Two vectors \(Y_i\) and \(Y_j\) span a plane in the signal space by a linear combination;
\[ \alpha \cdot Y_i + \beta \cdot Y_j \]
where \(\alpha, \beta\) are arbitrary complex numbers.
The spanned plane is represented by the vector
\[ Y(i,j) = \alpha(i,j) \cdot Y_i + \beta(i,j) \cdot Y_j \]
which maximizes the power ratio of the \(S_d\) component to that of the \(\{S_i\}\) components[1].
The above process is repeated to get the vector \(Y(1,2,\ldots,n)\) which represents the signal space formed by the auxiliary path signals.
Thus the subspace of the auxiliary paths receivers can be represented by a single vector as if being just a one-dimensional subspace.

5. Inclusion of Thermal noise into Signal Space

The main path signal X generally contains the desired signal Sd, the interference signals S1,S2,...,Sm and the thermal noise Nx.

The auxiliary paths signals \( \{Y_1,Y_2,...,Y_n\} \) each contain thermal noise Ni mainly originated at the receiver antenna and the front head Low Noise Amplifiers (LNA).

The main path and auxiliary paths signals are now expressed as follows;

\[
X = S_d + I_1 \cdot S_1 + I_2 \cdot S_2 +,...,+ I_m \cdot S_m + N_x
\]

\[
Y_i = D_i \cdot S_d + L_i1 \cdot S_1 + L_i2 \cdot S_2 +,...,+ L_i m \cdot S_m + N_i \quad (i = 1,2,...,n)
\]

How do we incorporate the thermal noise Ni into the signal space?

This problem is readily solved by TSS theorem as follows. The signal Yi is decomposed into two components;

\[
Y_i = (Y_i - N_i) + N_i = Y_{is} + N_i \quad (Y_{is} = Y_i - N_i)
\]

The first term contains all the signals Sd, S1, S2,...,Sm and the second term only the thermal noise. The thermal noise term contains no Sd, hence its tangent angle is zero. Then by TSS theorem the tangent angle of Y_{is} is equal to that of Yi. In another words the additional thermal noises expand the dimensions of the signal space but do not add any tangent square value of the auxiliary paths signal space.

The above description is readily depicted in the following figure.

Problem;

The representative vector of the above subspace (plane) spanned by Y_{is} (=Yi-Ni) and Ni is apparently Y_{is} itself which does not include the thermal noise Ni.

How can we get the subspace with the representative vector Yi including the noise?

Solution;

Let us first review Yi-Ni without the external noise.

\[
Y_{is} = D_i \cdot S_d + L_i1 \cdot S_1 + L_i2 \cdot S_2 +,...,+ L_i m \cdot S_m = D_i \cdot S_d + S_{yi}
\]

Where

\[
S_{yi} = L_i1 \cdot S_1 + L_i2 \cdot S_2 +,...,+ L_i m \cdot S_m
\]

Let us modify the above equation as follows:

\[
Y_{is} / \| S_{yi} \| = D_i / \| S_{yi} \| \cdot S_d + S_{yi} / \| S_{yi} \| = D_{ii} \cdot S_d + S_{yi} / \| S_{yi} \|
\]

Where
\[ D_{ii} = \frac{D_i}{\|S_i\|} \]

Then the tangent of \( Y_{is} \) is

\[ \tan^2(\theta_{Y_{is}}) = \frac{|D_{ii}|^2}{\|S_i\|^2} = \tan^2(\theta_{Y_{i}}) \quad \text{by TSS theorem} \]

Let us now express \( Y_i \) in the following form:

\[ Y_i = \alpha \cdot Y_{is}' + \beta \cdot Y_{in}' \]

\[ = D_{ii} \cdot (\alpha \cdot \cos(\phi) + \beta \cdot \sin(\phi)) \cdot \mathbf{S}_d + \alpha \cdot \frac{S_i}{\|S_i\|} + \beta \cdot \frac{\mathbf{N}_i}{\|\mathbf{N}_i\|} \]

where

\[ Y_{is}' = D_{ii} \cdot \cos(\phi) \cdot \mathbf{S}_d + \frac{S_i}{\|S_i\|} \]

\[ Y_{in}' = D_{ii} \cdot \sin(\phi) \cdot \mathbf{S}_d + \frac{\mathbf{N}_i}{\|\mathbf{N}_i\|} \]

The coefficients \( \alpha \) and \( \beta \) are determined to maximize the \( \tan^2(\theta_{Y_{i}}) \)

\[ \tan^2(\theta_{Y_{i}}) = \frac{|D_{ii} \cdot (\alpha \cdot \cos(\phi) + \beta \cdot \sin(\phi))|}{\|\alpha\|^2 + \|\beta\|^2} \]

To be maximized by coefficients \( \alpha \) and \( \beta \).

By differentiating

\[ \frac{\partial}{\partial \alpha} (\tan^2(\theta_{Y_{i}})) = 0 \]
\[ \frac{\partial}{\partial \beta} (\tan^2(\theta_{Y_{i}})) = 0 \]

We get

\[ \alpha = \cos(\phi) \]
\[ \beta = \sin(\phi) \]

And

\[ Y_i = D_{ii} \cdot S_d + \cos(\phi) \cdot \frac{S_i}{\|S_i\|} + \sin(\phi) \cdot \frac{\mathbf{N}_i}{\|\mathbf{N}_i\|} \]

It is obvious that

\[ \tan^2(\theta_{Y_{i}}) = \frac{|D_{ii}|^2}{\|S_i\|^2} = \tan^2(\theta_{Y_{is}}) \]

The \( \cos(\phi) \) and \( \sin(\phi) \) are directional cosines of the vector \( Y_i \) against the normalized vectors \( S_i / \|S_i\| \) and \( \mathbf{N}_i / \|\mathbf{N}_i\| \) on the \( S_i \times \mathbf{N}_i \) plane. The case \( \phi = 0 \) corresponds to the situation of no thermal noise. The case \( \phi = \pi / 2 \) corresponds to the situation where \( Y_i \) contains no \( S_i \) but only \( \mathbf{N}_i \) and \( S_d \).

The receiver's objective is to collect the interference signals \( S_i \) hence the signal-to-noise power ratio \( (S/N)_{ii} \) is

\[ (S/N)_{ii} = \frac{|\cos(\phi) \cdot \frac{S_i}{\|S_i\|}|^2}{|\sin(\phi) \cdot \frac{\mathbf{N}_i}{\|\mathbf{N}_i\|}|^2} = \cot^2(\phi) \]

Thus the inclusion of the thermal noise has been formulated as a rotation of the representative vector by an angle \( \phi = \arccot(\sqrt{(S/N)_{ii}}) \) around \( S_d \) axis.

The expansion of the dimension of the signal space by inclusion of the thermal noise is depicted in the following figure.
Inclusion of thermal noises

Reference

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