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電磁気学の基礎再考

市吉　修
†二十一世紀を楽しく生きよう会
〒252-0136 神奈川県相模原市緑区上九沢２３０－７
E-mail: osamu-ichiyoshi@muf.biglobe.ne.jp

あらまし

学校で教わる電磁気学は J.C.Maxwell が 1864 年に発表した 6 個の方程式を基礎としている。更に基礎的な法則として磁界の中を運動する荷電粒子に働く Lorentz 力がある。それは磁界 **B** の中を速度 **v** で運動する粒子にはベクトル積 **v** x **B** で表される電界が生じる事を意味する。Maxwell の方程式と Lorentz 電界は共に電磁気学の基礎として適宜応用されるが、両者の関係については解説される事が少ないようである。他方 1905 年に A.Einstein が発表した論文「運動体の電気力学」において Lorents 電界は相対性原理に基づく形で自然に導出された。即ち Lorentz 電界とは相対論的な現象であったのである。但しその電界はβ.**v** x **B** で表される。ここでβ=√(1- (v/c)^2)である。v は静止系に対する移動系の速度、c は光速である。即ち Lorentz 電界は相対論的な現象であり、通常の形式は v/c << 1 の場合に当たる近似的なものではないのかという疑問を筆者は持ってきた。その疑問は Maxwell の方程式から直接 Lorentz 電界を導く事と相対性理論の学習により氷解したので報告する。

キーワード
電磁気学、Maxwell の方程式、Lorentz 力、電磁誘導、特殊相対性理論理、サイクロトロン、相対論的質量、相対論的エネルギー

A Review of Basic Electromagnetism

Osamu Ichiyoshi
Human Network for Better 21 Century
230-7 Kamikuzawa, Midori-ku, Sagamihara City, Kanagawa prefecture , 252-0136 Japan
E-mail: osamu-ichiyoshi@muf.biglobe.ne.jp

Abstract

The electromagnetism taught at schools is usually based on the six equations published by J.C. Maxwell in 1864. There is another basic electromagnetic phenomenon called Lorentz' force, which states that to a moving particle at velocity **v** in a magnetic field **B** feels an electric field expressed by the vector product **v** x **B**. The two theories are applied properly to practical problems but their mutual relations seem rarely presented.
On the other hand the Lorentz' force was derived quite naturally in the paper "On electric dynamics of moving bodies" by A.Einsten in 1905 based on the relativity principle. However, the given formula is β.**v** x B, where β=√(1- (v/c)^2). **v** is the velocity of the moving particle, c is the speed of the light in vacuum. The author understood the Lorentz's field is a relativistic phenomenon but had the doubt that the usual expression is only approximate that holds only for v/c << 1.
The doubt is now cleared by deriving Lorentz formula from Maxwell's equations directly and further study of the relativistic theory as herein reported.

Keywords
Electromagnetism, Maxwell's equations, Lorentz's force, Special relativistic theory, electromagnetic induction

1. Maxwell's fundamental equations

The theoretical basis of electromagnetism was established by Maxwell's equations in 1864. In the following equations **E, H** are respectively Electric field and Magnetic field vectors as functions of space and time.

$$\nabla \times H = \mathbf{J} + \partial \mathbf{D}/\partial t \qquad (1\text{-}1)$$

$$\nabla \times \mathbf{E} = - \partial \mathbf{B}/\partial t \qquad (1\text{-}2)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (1\text{-}3)$$

$$\nabla \cdot \mathbf{D} = \rho \qquad (1\text{-}4)$$

where

$$\mathbf{B} = \mu \cdot \mathbf{H} \qquad (1\text{-}5)$$

$$\mathbf{D} = \varepsilon \cdot \mathbf{E} \qquad (1\text{-}6)$$

J is Electric current density, **B,D** are respectively Magnetic flux density and Electric flux density vectors. μ 、 ε are each magnetic permeability and dielectric permittivity which are constants in the isotropic media as treated in this paper.

ρ is electric charge density and is a scalar.

∇ is vector space differential operator ;

$$\nabla = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y + \mathbf{k}\partial/\partial z \qquad (1\text{-}7)$$

Where **i, j, k** are unit vectors in (x, y, z) directions.

Eq(1-1) is Ampere's law that tells how electric currents generate magnetic field. $\partial \mathbf{D}/\partial t$ has a similar property as electric current hence called Displacement current.

Eq(1-2) is Faraday's Electro-magnetic induction law that tells how time change of magnetic flux density **B** generates electric field.

Eq(1.4) is Gauss' law that tells about the relation between the electric field and charges.

Eq(1.3) tells there is no magnetic charge.

2. Lorentz's force

In addition to the Maxwell equations there is a basic term called Lorentz's force..

In an electric field **E** and magnetic field **B,** a particle with electric charge q and moving at velocity **v** is affected with the force **f**

$$\mathbf{f} = q.(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The above equation tells that the vector product $\mathbf{v} \times \mathbf{B}$ is equivalent to the electric field. A fundamental derivation of the formula was achieved by A. Einstein in his paper titled "Electrodynamics of moving bodies".[1] Einstein named it electro-motive force that a moving body running in a magnetic field feels an electric field.

In this section we will try to derive Lorentz's formula from Maxwell equations.

Suppose we have an electric charge q moving at velocity **v** in a magnetic field **B**.

We will time integrate the equation of electro-magnetic induction;

$$\int \nabla \times \mathbf{E}\, dt = - \mathbf{B}$$

then

$$q.\mathbf{v} \times \mathbf{B} = - q \int \mathbf{v} \times (\nabla \times \mathbf{E})\, dt$$

$$= q \int [\, (\mathbf{v}.\nabla)\, \mathbf{E} \;-\; \nabla(\mathbf{v}.\, \mathbf{E})\,]\, dt$$

$$= \int [\, (\mathbf{v}.\nabla)\, \mathbf{f} \;-\; \nabla(\mathbf{v}.\, \mathbf{f})\,]\, dt$$

Where $\mathbf{f} = q.\mathbf{E}$ is the force the electric field applies on the electrical charge.

On the other hand the force can be expressed mechanically with m; the mass of the particle.

$$\mathbf{f} = q.\mathbf{E} = m\, .\, \partial \mathbf{v}/\partial t$$

then

$$(\mathbf{v}.\nabla)\, \mathbf{f} = m.(v.\nabla)\, \partial \mathbf{v}/\partial t = 0$$

Let us now look at $\nabla(\mathbf{v}.\, \mathbf{f})$. The part $(\mathbf{v}.\, \mathbf{f})$ is the rate of mechanical work, hence its time integration is the work or energy.

There is no loss of energy in the vacuum, hence it must be stored to form a potential ϕ.

In summary,

$$q.(\mathbf{v} \times \mathbf{B}) = - \nabla \phi$$

that is, $q.(\mathbf{v} \times \mathbf{B})$ is a force given by gradient of a potential.

Thus $\mathbf{v} \times \mathbf{B}$ is an electric field generated by the magnetic field to a moving body.

<u>Example1</u>

Suppose a particle with mass m electrically charged with q comes into a uniform magnetic field **B** with velocity **v.** The Lorentz force q.(**v**×**B**) is perpendicular to both **B** and **v.** Thus the particle moves in a circular mode around a central point, equivalent to a movement caused by a centripetal force. This is applied to cyclotron to accelerate charged particles.

Let r be the radius of the circle. The centripetal and centrifugal forces balance;

m.**v**^2/r = q.**v. B**

The radius r is;

r = (m.v) / (q.B)

The circular motion is equivalent to one where the centripetal force is provided by an electrical charge q' fixed at a point;

-(q.q') /(r^2) = q.v. B

Thus

q' = -(m.v / q)^2 / B

The equivalent potential is;

ϕ = q' / r

Example 2

Let us study the case a rectangular shaped conductor is placed in a uniform magnetic field **B**. The conductor rectangle can mechanically turn around the axis as depicted in the following figure.

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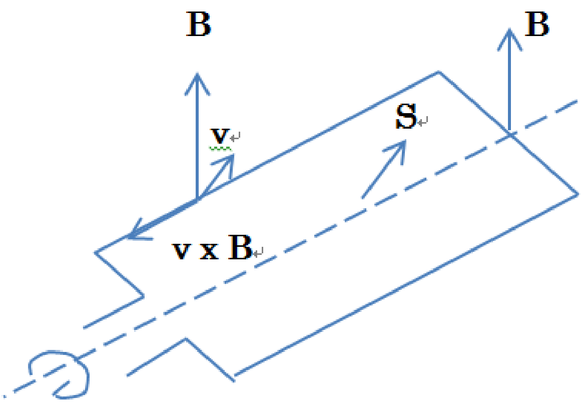

Solution by Lorentz's field

The upper side of the rectangle generates Lorentz field **E** as shown in the figure and the lower side in the opposite direction. By integration of the field along the conductor, one gets the voltage V at the terminal

∫**E** ds = v.B.2L.sin(θ)

Where 2L is the length of the conductor and θ is the angle between **v** and **B.**

Let r be the radius of the above rotator and ω be angular velocity.

Since v = r. ω

then

V = 2r. ω .B.L.sin(θ) (θ = ω .t t ;time)

Solution by Faraday's electromagnetic induction

The basic equation

∇x **E** = - ∂**B**/∂t

is surface integrated;

∫ ∇x **E** d**S** = - ∫ ∂**B**/∂t d**S**

= - [∂/∂t] ∫ **B** d**S**

= - [∂/∂t]2rL.cos(θ)

The left side by Stokes' theorem terns to the line integral;

∫ ∇x E dS = ∫ E dl = -V

Therefore

V = [∂/∂t]2rLcos(θ)

= 2r.ω.B.L.sin(θ)

Examination of the examples

Example 2 can be analyzed by either Lorentz's or Faraday's methods to give the same result. The rotator works as a dynamo if it is mechanically turned by external torque, or it works as a motor if the voltage V is externally applied to give the current that then generates the torque based on Lorentz' force.

Example 1 is not so clear. The point charge moves in a circle. If the circle is small and the magnetic field **B** is sufficiently uniform, then its time derivative must be zero hence no electrical field be generated by electromagnetic induction, apparently contradictory to the facts.

3. Special Relativity theory

Another derivation of Lorentz's field is obtained by special theory of relativity[1] presented by A. Einstein in 1905.

3-1 Special relativity theory of moving bodies

Principle of constant speed of light in vacuum

The theory is based on the experimental fact the speed of light in vacuum is constant regardless of the movement of the observers;

$c = 3 \times 10^8$(m/s)..

This is hard to grasp by particle images of light, but understandable by the wave theory of the light. According to Maxwell's theory the speed of light is given by

$c = 1/\sqrt{(\varepsilon \cdot \mu)}$

The dielectric permittivity ε and the magnetic permeability μ in the vacuum must remain the same regardless of the movement of the observer hence so be the speed of light.

Galilei Ttransformation

Suppose we have two observing systems K and k where k is moving against K at a constant speed v in the x direction.

Let us express the time and space coordinate of the point in system K and k by (t , x) and (τ , ξ).

The classical Galilei transformation is;

$\xi = x - v.t$

$\tau = t$

This is quite natural in daily life where the time and space are mutually independent. However, a body moving in k at speed w will be observed in K as moving at v + w which can exceed c, thus contradicts to the fact.

Lorentz Transformation

In order to meet the principle of constancy of the speed of light, the time-space coordinates of the point (t, x, y, z) in K and (τ , ξ , η , ζ) in k must follow the following Lorentz Transformation;

$$\xi = \beta .(x - v.t) \qquad (3.1\text{-}1)$$

$$\tau = \beta. (t - v/c^2.x) \qquad (3.1\text{-}2)$$

$$\eta = y \qquad (3.1\text{-}3)$$

$$\zeta = z \qquad (3.1\text{-}4)$$

where

$$\beta = 1/\sqrt{(1 - (v/c)^2)} \qquad (3.1\text{-}5)$$

Relativity principle

The reverse transformation must be obtained by replacing v with –v;

$$x = \beta .(\xi + v. \tau) \qquad (3.1\text{-}1)'$$

$$t = \beta .(\tau + v/c^2. \xi) \qquad (3.1\text{-}2)'$$

$$y = \eta \qquad (3.1\text{-}3)'$$

$$z = \zeta \qquad (3.1\text{-}4)'$$

Four dimensional continum

In the static coordinate system the time t and space (x, y, z) are independent but in the moving system observed from the static system the time τ and space (ξ, η, ζ) are not independent but form a four dimensional continuum (τ, ξ, η, ζ).

Slower flow of time in moving bodies

A clock placed at the origin of the moving system k; $\xi = 0$ shows the time;

$$\tau = 1/\beta .t = \sqrt{(1 - (v/c)^2)}\, t$$

longitudinal Shrinkage of moving bodies

The difference between two points ($\tau 1$, $\xi 1$) and ($\tau 2$, $\xi 2$) in the moving system k observed from the static system K is;

$$x2 - x1 = \beta .\{ (\xi 2 - \xi 1) + v. (\tau 2 - \tau 1) \}$$

For $\tau 2 - \tau 1 = 0$,

$$\xi 2 - \xi 1 = 1/\beta . (x2 - x1)$$

$$= \sqrt{(1 - (v/c)^2)} . (x2 - x1)$$

Thus occurs shrinkage of bodies in the direction of movement. Note a body is defined as a system where the time flows uniformly.

Quantities conserved in different coordinate systems

The following quantities are conserved between systems k and K.;

$$(c.t)^2 - (x^2 + y^2 + z^2) = (c. \tau)^2 - (\xi^2 + \eta^2 + \zeta^2)$$

(In static system K) (in moving system k)

In differential form;

Suppose the above equation holds for small coordinate changes;

$t \rightarrow t+\Delta t, \quad x \rightarrow x+ \Delta x, \quad y \rightarrow y+ \Delta y, \quad z \rightarrow z+ \Delta z$,等

Then the following differential forms of conservation holds.

$$(c.\Delta t)^2 - \{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\}$$
$$= (c.\Delta \tau)^2 - \{(\Delta \xi)^2 + (\Delta \eta)^2 + (\Delta \zeta)^2\}$$

3.2 Electrodynamics of moving bodies

The Maxwell's equations are expressed in the static coordinate system K (t, x, y, z). The electric field vector **E** has components (Ex, Ey, Ez) in (x, y, z) directions.

Likewise for **D, B, H**, etc.

Relativity principle in vacuum

Let us express the electric field vector as

$$\mathbf{E}' = (E\xi', \quad E\eta', \quad E\zeta')$$

and differential operator as;

$$\nabla' = \mathbf{i}\partial/\partial\xi + \mathbf{j}\partial/\partial\eta + \mathbf{k}\partial/\partial\zeta$$

in the moving system k (τ, ξ, η, ζ).

By the relativity principle the same form of equations must hold in the moving system k as in the static system K.;

$$\nabla' \times \mathbf{H}' = \mathbf{J}' + \partial\mathbf{D}'/\partial\tau \qquad (3.2\text{-}1)$$
$$\nabla' \times \mathbf{E}' = -\partial\mathbf{B}'/\partial\tau \qquad (3.2\text{-}2)$$
$$\nabla' \cdot \mathbf{B}' = 0 \qquad (3.2\text{-}3)$$
$$\nabla' \cdot \mathbf{D}' = \rho' \qquad (3.2\text{-}4)$$

Transformation of electromagnetism in vacuum

In vacuum,

$$\mathbf{J} = 0, \quad \rho = 0 \qquad (3.2\text{-}5)$$

The relativity principle tells the following relationship must hold.

$$E\xi' = Ex \qquad (3.2\text{-}6)$$
$$E\eta' = \beta(Ey - v.Bz) \qquad (3.2\text{-}7)$$
$$E'\zeta = \beta(Ez + v.By) \qquad (3.2\text{-}8)$$

$$B'\xi = Bx \qquad (3.2\text{-}9)$$
$$B'\eta = \beta(By + v/c^2.Ez) \qquad (3.2\text{-}10)$$
$$B'\zeta = \beta(Bz - v/c^2.Ey) \qquad (3.2\text{-}11)$$

The equations tell

(1) The longitudinal field components (in direction of the movement) are unchanged,

(2) The transversal components (in perpendicular to the movement direction) have the following relationship

$$\mathbf{E}'t = \beta.(\mathbf{E}t + \mathbf{v} \times \mathbf{B}t)$$

Namely, except for the coefficient $\beta = 1/\sqrt{(1 - (v/c)^2)}$, it gives the Lorentz field as the vector product of **v** and **B**. Thus we can say the Lorentz field is a relativistic phenomenom.

Furthermore for the magnetic field;

$$\mathbf{B}'t = \beta.(Bt - \mathbf{v} \times \mathbf{E}t / c^2)$$

In conclusion,

(3) The field components remain the same in the longitudinal direction but the transversal components are combined by a sort of Lorentz transformation. Thus electric and magnetic fields are not independent hence must be treated as combined electromagnetism.

3.3 Movements of electron in electromagnetic field

In the moving system k attached to an electron the following equations of motion must hold;

$$m.[(d/d\tau)^2]\xi = e.E\xi' = e.Ex$$
$$m.[(d/d\tau)^2]\eta = e.E\eta' = e.\beta.(Ey - v.Bz)$$
$$m.[(d/d\tau)^2]\zeta = e.E\zeta' = e.\beta.(Ez + v.By)$$

where "m " is the mass and "e" the electric charge of the electron.

What do we observe this from the static field K?

Local time of the electron

The motion of the electron is not generally constant velocity linear movement but follows a bent curve, Therefore we must use the conserved quantity in differential form;

$$(c.\Delta t)^2 - \{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\}$$
$$= (c.\Delta\tau)^2 - \{(\Delta\xi)^2 + (\Delta\eta)^2 + (\Delta\zeta)^2\}$$

Since the electron is always at the origin of system k;

$$(\Delta\xi)^2 + (\Delta\eta)^2 + (\Delta\zeta)^2 = 0$$

then

$$d\tau / dt = \sqrt{(1 - (\mathbf{v}/c)^2)}$$

$= \sqrt{(1 – (d\mathbf{r}/dt/c)^2)}$

$= 1/\beta$

where

$\mathbf{v} = d\mathbf{r}/dt$

$\mathbf{r} = (x, y, z)$

The " τ " is the **local time** of the moving electron.

Equations of the motion observed from static system K

For the longitudinal direction both time and space shrink by the same rate hence

$d\xi/d\tau = dx/dt$

therefore

$[(d/d\tau)^2]\xi = [d/d\tau][dx/dt]$

$= [d/dt][dx/dt].[dt/d\tau]$

$= \beta.[(d/dt)^2]x$

For the transversal components;

$[(d/d\tau)^2]\eta = \beta^2.[(d/dt)^2]y$

$[(d/d\tau)^2]\zeta = \beta^2.[(d/dt)^2]z$

Thus we ge the equations of motion observed in static system K;

$m.\beta,[(d/dt)^2]x = e.Ex$

$m.\beta,[(d/dt)^2]y = e.(Ey – v.Bz)$

$m.\beta.[(d/dt)^2]z = e.(Ez + v.By)$

3.4. Relativistic momentum and mass

The above analysis tells

(1) The forces applied to the electron are the usual electrical and Lorentz' field forces.

(2) The mass increases by the rate $\beta = 1/\sqrt{(1 – (v/c)^2)}$ where v is the speed of the electron observed in static system K.

(3) The relativistic mass is experimentally proven and gives a direct ground why no matter can reach the speed of light.

Relativistic energy

Since x is no special direction we will use $\mathbf{r} = (x, y, z)$ to represent the location of the electron.

$d\mathbf{r}/dt = (d\mathbf{r}/d\tau).(d\tau/dt)$

$= (d\mathbf{r}/d\tau).\sqrt{(1 - ((d\mathbf{r}/dt)/c)^2)}$

$= (d\mathbf{r}/d\tau) / \sqrt{(1 + ((dr/d\tau)/c)^2)}$

On the other hand

$d\tau/dt = \sqrt{(1 - ((d\mathbf{r}/dt)/c)^2)}$

$= 1/\sqrt{(1 + ((d\mathbf{r}/d\tau)/c)^2)}$

Therefore

$m.c^2 [d/dt] (dt/d\tau)]$

$= m.c^2 .[d/dt] \sqrt{(1 + ((d\mathbf{r}/d\tau)/c)^2)}$

$= (d\mathbf{r}/d\tau) / \sqrt{(1 + ((d\mathbf{r}/d\tau)/c)^2)} . m.[d/dt] (d\mathbf{r}/d\tau)$

$= d\mathbf{r}/dt. \mathbf{F}$

$= \mathbf{v.F}$

where

$\mathbf{F} = m.[d/dt] (d\mathbf{r}/d\tau)$

$= \beta m. [(d/dt)^2]\mathbf{r}$

is force.

Furthermore

$\mathbf{v.F} = dE/dt$

is the time rate of Energy E of the moving particle.

Thus

$E = m.c^2.(dt/d\tau)$

$= m.c^2 /\sqrt{(1 - (v/c)^2)}$

$= m.c^2 + 1/2\, m.v^2 \qquad (v/c << 1)$

The first term is the static energy and the second term is the kinetic energy of the particle.

The static energy tells the equivalence of mass and energy.

References

[1] Tatsuo Uchiyama Einstein's Relativity Theory

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