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複素ベクトル・スカラー環代数論

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二十一世紀を楽しく生きよう会

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あらまし

宇宙ロケットや衛星等の飛行体の制御には飛行体に固定した機体座標系と宇宙に固定した慣性座標系との持続的な変換が必要不可欠である。座標変換は機体に固定した機体ベクトルを測定系より与えられる回転軸ベクトルの回りに与えられた角度だけ回転させる事により実行される。その優美な計算方法として四元数を用いる方法がある。四元数は虚数を三次元に拡張したものと見なせるが､虚数を用いずにベクトルだけを用いる方法は無いであろうか。四元数の定義は三次元ベクトルのベクトル積に類似している事からベクトル演算としてベクトル・スカラー積(x)なるものを用いればよろしい。それはベクトル **u,v** に対して **u** (x) **v** = **u** x **v** – (**u.v**)として定義される。但し **u** x **v** は通常のベクトル積あるいは外積、(**u.v**)はスカラー積あるいは内積である。三次元空間の基底ベクトルを **i, j, k,**とするとベクトル **u** = u1.**i** + u2.**j** + u3.**k** = (u1, u2, u3) と表現できるが、これにスカラー成分を加えて[u] = (u0, u1, u2, u3)なる四元ベクトル代数系としてベクトル・スカラー系を定義するとそれは四元数に等価な環を成す。更に四元ベクトルの成分 u0, u1, u2, u3 は複素数であり得るからそれは複素ベクトル・スカラー系となる。複素ベクトル・スカラー系は内積と外積その他のベクトル演算により四元数より遥かに簡潔に計算を行う事ができる。
ベクトル･スカラー積は単位大きさの任意の実ベクトル l に対して l (x) l = -1 となるので虚数に類似の性質を持っている。これにより cos(θ) + **l**.sin(θ) = e^(**l.** θ)としてベクトル・スカラー(VS)の極座標表示が可能となる。
本稿においては VS の逆数を定義して加減乗除の演算法を確立し、それに基づいて、累乗、累乗根、指数関数、対数関数、三角関数等を定義し、一次方程式、及び二次方程式の解法などを示す。

キーワード　宇宙ロケット、座標変換、機体座標、慣性座標、ベクトルの回転、四元数、四元ベクトル、回転角、回転軸、ベクトル・スカラー 積、外積、内積

A Complex Vector-Scalar Ring Theory

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Abstract

The quaternion is an expansion of complex number to three dimensions of imaginary numbers. It is a useful tool in calculating rotation of vectors around a given axis in the three dimensional space. The imaginary numbers in quaternion can be replaced with real vectors in the three-dimensional space to give a Vector-Scalar (VS). The set of whole vector-scalars is algebraically equivalent to that of quaternions; they form rings. The transition is made by a definition of vector-scalar product (x) as follows. For vectors **u** and **v, u** (x) **v** = **u** x **v** – (**u.v**), where **u** x **v** and (**u.v**) are respectively normal vector product and scalar product. For any vector **l** with unit length, **l** (x) **l** = -1, which is similar to the imaginary number i. In fact the following formula e^(**l** θ) = cos(θ) + **l**.sin(θ) can be defined just as Euler's formula in complex number theory. The coefficients in VS can take complex values to achieve a fundamental unification of vectors and complex numbers. Functions in VS domain can be defined in much the same manners as in complex plane enabling to solve wide ranges of vectors and scalars problems.

Keywords　coordinates, conversion, rotation, vector, scalar, vector-scalar, quaternion, ring, vector-scalar product,

1. Rotation of vector in space

Let a vector **r** rotated around an axis **e** by an angle θ to get vector **r**'. Here **e** is a unitary vector in three dimensional space ; (**e.e**) = 1.

The vector **r** can be expressed in the following form

$$\mathbf{r} = (\mathbf{r.e})\mathbf{e} \quad + \quad \mathbf{e} \times (\mathbf{r} \times \mathbf{e}) \qquad (1.1)$$

The first term is the axis component and the second term is the transversal component.

The rotated vector **r**' is given by;

$$\mathbf{r}' = \mathbf{e} \times (\mathbf{r} \times \mathbf{e}).\cos(\theta) + (\mathbf{e} \times \mathbf{r}).\sin(\theta) + (\mathbf{r.e})\mathbf{e}$$
$$= (\mathbf{e} \times \mathbf{r}) \times \{\mathbf{e}.\cos(\theta) + \sin(\theta)\} + (\mathbf{r.e})\mathbf{e} \qquad (1.2)$$

The rotation process can be repeated to get **r**" from **r**'. But the relation between **r**" and **r** is complicated.

2. Vector Rotation formula with Quaternion

2.1. Quaternion

A quaternion z is defined as follows;

$$z = a + ib + jc + kd \quad = (a,b,c,d)$$

where a,b,c,d are real numbers and i, j, k are imaginary numbers. The operations of the imaginary numbers are;

$i^2 = j^2 = k^2 = \quad i.j.k = -1$
$i \cdot j = -j \cdot i = k$
$j \cdot k = -k \cdot j = i$
$k \cdot i = -i \cdot k = j$

In the Quaternin (a,b,c,d), a is **real part** and (b,c,d) is **imaginary part.**

Ring

Quaternions z ,u, w have the following properties;

z.u =/= u.z ;Commutative law not held
(z.u).w = z.(u.w) = z.u.w ; Associative law held
z.(u+w) = z.u+z.w
(z+u).w = z.w+u.w; Distributive law held

Thus the set of whole quaternions forms a Ring.

Quaternion Conjugate

For quaternion z = a + ib + jc + kd

Its conjugate is defined as follows

$$z^* = a - ib - jc - kd$$

Real part snd imaginary part

Definitions;

Real part; $a = (z + z^*)/2$
Imaginary part; $(b, c. d) = (z - z^*)/2$

Conjugate of product of quaternions

For quaternions z and w, the following holds;

$$(z . w)^* = w^* . z^*$$

Absolute value of quaternion

The product of z and z* gives;

$$z.z^* = a^2 + b^2 + c^2 + d^2 = |z|^2$$

Where |z| is the absolute value of z.

2.2 Quaternion formula on rotation of vectors

In equations (1.1) and (1.2), let the axis vector **e** expressed in quaternion as;

$$e = i.l + j.m + k.n = (0, l, m, n)$$

where l,m,n are direction cosines for 3 coordinates.

$$l^2 + m^2 + n^2 = 1$$

Let the vectors **r** and **r**' expressed as quaternions;

$r = (0, x, y, z)$
$r' = (0, x', y', z')$

Then the rotation of the vector **r** is expressed in quaternions products as follows;

$$r' = T . r . T^* \qquad (2.2\text{-}1)$$

where

$$T = \cos(\theta/2) + \sin(\theta/2).e \qquad (2.2\text{-}2)$$

The equivalence of equations (1.2) and (2.2-1,2) can be proved by direct calculation, which is fairly cumbersome.

3. Vector-Scalar Ring

3.1. Basis vectors

The axis vector **e** is expressed by the directional cosines l, m, n and the coordinate unit vectors **i,j,k** ;

$\mathbf{e} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$
$(\mathbf{e.e}) = l^2 + m^2 + n^2 = 1$

The basis vectors have the following two different kinds of products;

Scalar or Inner products;

$(\mathbf{i.i}) = (\mathbf{j.j}) = (\mathbf{k.k}) = 1$
$(\mathbf{i.j}) = (\mathbf{j.i}) = 0$

(**j.k**) = (k.**j**) = 0
(**k.i**) =(i.**k**) = 0

Vector or Outer products;

i x i = j x j = k x k = 0
i x **j** = -**j** x **i** = **k**
j x **k** = -k x **j** = **i**
k x **i** = -**I** x **k** = **j**

3.2. Definitions of Vector-Scalar

For real numbers a,b, vector-scalar is defined by;

[**z**] = a + b.**e** = (a, b.l, b.m, b.n)

Scalar part and vector part

In the above formula a is Scalar Part and b Vector Part of vector-scalar [z]..

Vector-Scalar product

For vectors **u, v** the vector-scalar product is defined **;**

u (x) **v** = **u** x **v** – (**u, v)**
(Outer product) (inner product)

Multiplication (x) is a simple scalar multiplication unless both operands are vectors.

For the axis vector **e**

e (x) **e** = -1

Scalar and Vector products

Conversely the Scalar and Vector Products can be defined by Vector-Scalar product.

(**u.v**) = - { **u** (x) **v** + **v** (x) **u** } / 2
u x **v** = { **u**(x)**v** - **v**(x)**u** } / 2

Equivalence of Vector-Scalar and Quaternion

The above definitions tell the whole sets of vector-scalars and quaternions form equivalent algebraic entity, i.e. they form Rings.

3.3 Vector conjugate

The vector conjugate of vector-scalar

[z] = a + b.**e**

is defined as follows:

$[z]^{vc}$ = a - b.**e**

Then the following product gives

[z] (x) $[z]^{vc}$ = (a + b**e**) (x) (a – b**e**)
= a^2 + b^2
= |[z]|^2

Where |[z]| is the absolute value of [z].

Scalar Part and Vector Part

Definitions:

Scalar Part;

a = { [z] + $[z]^{vc}$ }/2
= Sc[z]

Vector Part;

b = { [z] - $[z]^{vc}$ }/(2 **e**)
= Vc[z]

Where 1/**e** = -**e,** since

(1/**e**) (x) **e** = -**e** (x) **e** = 1.

3.4. Vector-scalars with different axis vectors

Let axis vectors; **e, f** form vector-scalars [z], [w]

[z] = a + b**e**
[w] = c + d**f**

Then

[z](x)[w] =(a+b**e**) (x) (c+d**f**)
= a.c – b.d.(**e.f**) + b.c.**e** + a.d.**f** + b.d.**e** x **f**

On the other hand

[w](x)[z] =(c+d**f**) (x) (a+b**e**)
= a.c – b.d.(**e.f**) + b.c.**e** + a.d.**f** + b.d.**f** x **e**

Hence,

[z](x)[w] =/= [w](x)[z]

The commutative law does not hold unless axis vectors are identical; **e = f.**

Conjugate of Products of VS

$\{[z] (x) [w]\}^{vc}$ = $[w]^{vc}$ (x) $[z]^{vc}$

Absolute value of Products

|[z] (x) [w]| = |[z]|・|[w]|

3.5. Polar representation of Vector-Scalar

Let a vector-scalar [x] be expressed as follows;

[x] = x0 + **x**
= (x0, x1, x2, x3)

Where x0 is the scalar part

and

x = x1.**i** + x2.**j** + x3.**k**

is the vector part.

The absolute values of vector-scalar and vector are;

|[x]| = SQR{(x0)ˆ2 + |**x**|ˆ2 }
|**x**| = SQR{ x1ˆ2 + x2ˆ2 + x3ˆ2}

Then,

[x] = x0 + **x**
=|[x]|.{cos(θx) + sin(θx).1x }
=|[x]|.e^(θx . **l**x)

where

$\cos(\theta x) = x0 / |[x]|$

$\sin(\theta x) = |\mathbf{x}| / |[x]|$

$1x = \mathbf{x} / |\mathbf{x}|$

Proof;

Let vector scalar [T] be

$[T] = \cos(\theta) + \sin(\theta).\mathbf{l}$

Differentiation of [T] by variable θ gives

$[d/d\theta][T] = -\sin(\theta) + \cos(\theta).\mathbf{l}$

$= \mathbf{l}\ (x)\ [T]$

Hence

$[T] = e\verb|^|(\mathbf{l}\theta) = \cos(\theta) + \sin(\theta).\mathbf{l}$

3.6 Vector-Scalar representation of vector rotation in three dimensional space

The equations (2.2-1,2) in quaternion form can be now re-written in vector-scalar form;

$\mathbf{r}' = \{\cos(\theta/2) + \sin(\theta/2).\mathbf{e}\}$

$(x)\ \mathbf{r}$

$(x)\ \{\cos(\theta/2) + \sin(\theta/2).\mathbf{e}\}^{vc}$

$= e\verb|^|(\theta/2.\mathbf{e})\ (x)\ \mathbf{r}\ (x)\ e\verb|^|(-\theta/2.\mathbf{e})$

Its equivalence with equation (1-2) can be proved by direct calculation using vector analysis.

3.7. Complex Vector-Scalar

Formulation;

VSR [z] is expressed as

$[z] = z0 + \mathbf{z}$

$= z0 + z1.\mathbf{i} + z2.\mathbf{j} + z3.\mathbf{k}$

Where the coefficients {zn; n = 0,1,2,3 } can take complex values in general;

$zn = xn + yn.i \qquad (i\verb|^|2 = -1)$

where {xn} are the real parts and {yn} are imaginary parts

Vector conjugate and complex conjugate

For VS

$[z] = z0 + \mathbf{z},$

Vector Conjugate is defined as;

$[z]^{vc} = z0 - \mathbf{z}$

On the other hand,

Complex Conjugate is defined as ;

$[z]^{cc} = z0^{cc} + \mathbf{z}^{cc}$

$= x0 + x1.\mathbf{i} + x2.\mathbf{j} + x3.\mathbf{k}$

$- i.\{y0 + y1.\mathbf{i} + y2.\mathbf{j} + y3.\mathbf{k}\}$

Products of Vector-Scalar with its vector and complex conjugate

$[z]\ (x)\ [z]^{vc,cc} = |z0|\verb|^|2 + |\mathbf{z}|\verb|^|2$

$- z0.\ \mathbf{z}^{cc} + z0^{cc}.\mathbf{z} - \mathbf{z}\ x\ \mathbf{z}^{cc}$

Note the scalar part is non-negative real value and the vector part is purely imaginary;

$Re\{[z](x)[z]^{vc,cc}\} = Sc\{[z](x)[z]^{vc,cc}\}$

$= |z0|\verb|^|2 + |\mathbf{z}|\verb|^|2$

$Im\{[z](x)[z]^{vc,cc}\} = Vc\{[z](x)[z]^{vc,cc}\}$

$= - z0.\ \mathbf{z}^{cc} + z0^{cc}.\mathbf{z} - \mathbf{z}\ x\ \mathbf{z}^{cc}$

Norm of complex VS

Square root of the above scalar part is called the norm of [z];

$\|[z]\| = SQR(|z0|\verb|^|2 + |\mathbf{z}|\verb|^|2)$

Polar representation of complex VS

$[z] = z0 + \mathbf{z}$

$= \|[z]\|.(\ |z0|/\|[z]\|.z0/|z0|$

$+ |\mathbf{z}|/\|[z]\|.\mathbf{z}/|\mathbf{z}|\ )$

$= \|[z]\|.\{\cos(\theta z).\ z0/|z0| + \sin(\theta z).1z\}$

where

$\cos(\theta z) = |z0|/\|[z]\|$

$\sin(\theta z) = |\mathbf{z}|/\|[z]\|$

$1z = \mathbf{z}/|\mathbf{z}|$

Separating the real and imaginary parts;

$[z]/\|[z]\|$

$= \cos(\theta z).\ z0/|z0| + \sin(\theta z).1z$

$= \cos(\theta z).\ x0/|z0| + \sin(\theta z).\mathbf{x}/|\mathbf{z}|$

$+ i.\{\cos(\theta z).y0/|z0| + \sin(\theta z).\mathbf{y}/|\mathbf{z}|\}$

$= \cos(\theta z).\cos(\phi) + \sin(\theta z).\cos(\Psi)1x$

$+ i.\{\cos(\theta z).\sin(\phi) + \sin(\theta z).\ \sin(\Psi)1y\}$

where

$\cos(\phi) = x0/|z0|$

$\sin(\phi) = y0/|z0|$

$\cos(\Psi) = |\mathbf{x}|/|\mathbf{z}|$

$\sin(\Psi) = |\mathbf{y}|/|\mathbf{z}|$

$1x = \mathbf{x} / |\mathbf{x}|$

$1y = \mathbf{y} / |\mathbf{y}|$

4. Operations of Vector-Scalars

4.1 Ring

No commutative law does not hold ;

[x] (x) [y] =/= [y] (x) [x];

Associative law holds;

([x] (x) [y]) (x) [z] = [x] (x) ([y] (x) [z]);

Distributive law holds;

[x] (x) ([y] + [z]) = [x] (x) [y] + [x] (x) [z]

([x] + [y]) (x) [z] = [x] (x) [z] + [y] (x) [z] ;

Thus Vector-Scalars form a ring ; VSR.

4.2. Inverse element

For Vector-Scalar [z]= z0 + **z**

[z] (x) $[z]^{vc}$ = (z0 + **z**) (x) (z0 – **z**)

= z0^2 + (**z.z**)

= |[z]|^2

Hence the inverse of [z] shall be

1/[z] = $[z]^{vc}$ / |[z]|^2

where

|[z]|= SQR(z0^2 + (**z.z**))

Note the absolute value can take complex value in general.

For the inverse element the commutative law holds;

1/[z] (x) [z] = [z] (x) 1/[z] = 1

Especially for the case z0 = 0, the inverse of the vector is

1/**z** = - **z** / (**z.z**)

4.3. Multiplication

Let multiplication of VS [x] and [y] gives [z].

[x] (x) [y] = (x0 + **x**) (x) (y0 +**y**)

= x0.y0 – (**x.y**) + x0.**y** + y0.**x** + **x**×**y**

In polar forms;

[x] = ‖[x]‖. e^(θx. 1x)

= ‖[x]‖. (cos(θx) + sin(θx). 1x)

and

[y] = ‖[y]‖. (cos(θy) + sin(θy). 1y)

[z] = ‖[z]‖. (cos(θz) + sin(θz). 1z)

Then for

[z] = [x] (x) [y]

In norms or amplitude;

‖[z]‖ = ‖[x]‖. ‖[y]‖

In phases;

cos(θz) = cos(θx). cos(θy)

− sin(θx). sin(θy). (1x. 1y)

sin(θz). 1z = cos(θx). sin(θy). 1y

+ cos(θy). sin(θx). 1x

+ sin(θx). sin(θy). (1x) x (1y)

Thus we have established formulae for addition, subtraction, multiplication and division.

5. Functions of Vector-Scalars

5.1 Powers and Roots

Powers:

In the above multiplication formula, let [x] = [y], then

[x] ^2 = ‖[x]‖^2

. (cos(2θx) + sin(2θx). 1x)

= ‖[x]‖^2. e^(2θx. 1x)

In general for integer n:

[x] ^n = ‖[x]‖^n . e^(n. θx. 1x)

Roots

For VS [b]

[b] = b0 + **b** = ‖[b]‖. e^(θb. 1b)

Let [x] be

[x]^2 = [b]

Then [x] shall be square root (SQR)of [b]

‖[x]‖^2. e^(2θx. 1x)

= ‖[b]‖. e^(θb. 1b)

= ‖[b]‖. e^((θb+2π). 1b)

Hence

‖[x]‖ = SQR(‖[b]‖)

θx = θb/2, θb/2 +π

1x = 1b

Namely we have two roots;

[x]= +,- SQR(‖[b]‖). e^(θb/2. 1b)

In general we have n roots for [x]^n =1.

5.2. Exponential function

For real number p

p^[x] = p^(x0 + **x**)

= p^x0. e^(log(p).x.**l**x)

= p^x0.(cos(log(p).x) + sin(log(p).x). **l**x)

where

x = |**x**|

1x = **x** / x

For real numbers p, q

(p.q)^[x] = p^[x] (x) q^[x]

On the other hand

[x]^p = { ‖[x]‖.e^(θx.1x)}^p

= ‖[x]‖^p .e^(p.θx.1x)

For different [x], [y]

([x] (x) [y])^p =/= [x]^p (x) [y]^p

p^([x]+[y]) =/= p^[x] (x) p^[y]

For Vector-Scalars with different axis vectors the exponential laws do not hold, therefore, exponential functions cannot be well defined. For the VSs with the same axis vector, they form a "field" as the commutative law also holds so all the functions can be well defined.

5.3 Logarithmic functions

For VS

[x] = ‖[x]‖.e^(θx.1x)

= ‖[x]‖.e^((θx + 2πn).1x) (n; integer)

The logarithm shall be

Log([x]) = log(‖[x]‖)+ (θx+2πn).1x

5.4 Trigonometric functions

are defined with the exponential functions.

cosh([x])= {e^[x] + e^(-[x])}/2

sinh([x])= (e^[x] - e^(-[x])}/2

cos(x) = (e^(x.1x) + e^(-x.1x))/2

sin(x) = (e^(x.1x) - e^(-x.1x))/(21x)

where

1x = **x**/x

1x (x) 1x = -1

Trigonometric functions of complex VSR

cosh([z]) = cosh(z0 + **z**)

= cosh(z0).cos(z) +1z.sinh(z0).sin(z)

sinh([z]) = sinh(z0 + **z**)

= sinh(z0).cos(z) +1z.cosh(z0).sin(z)

For complex [z] composed of real and imaginary parts;

z0 = x0 + y0.i

z = **x** + **y**.i

The unit vector 1z can be decomposed as

1z = **z**/z = x/z.**x**/x + y/z.**y**/y.i

= cos(φ).1x + i.sin(φ).1y

Where

z = |**z**| = SQR(x^2 + y^2)

x = |**x**| = SQR(x1^2 + x2^2 + x3^2)

y = |**y**| = SQR(y1^2 + y2^2 + y3^2)

also

sinh(z0) = sinh(x0).cos(y0) + i.sin(y0).cosh(x0)

cosh(z0) = cosh(x0).cos(y0) + i.sin(y0).sinh(x0)

6. Equations of Vector-Scalars

Equations containing unknown VS; [x] = (x0, x1, x2, x3) can be solved in principle by decomposing the equations into the elements of VS. However, it is equivalent to calculation of quaternions, which is usually quite cumbersome.

Calculations with VSs can be made more simply and understandably utilizing various vector analysis tools. In addition complex VSR expands the ranges of problems that can be handled.

6.1. Linear equations

[1] Basic equations

[a] (x) [x] = [b]

The solution is;

[x] = 1/[a] (x) [b]

[2] Linear equations with unknown Vectors

For the equation with unknown vector **y** and given vectors **a** and **b;**

a x **y** = **b** (6.1)

In order to form Vector-Scalar equation we set an auxiliary equation

(**a.y**) = λ (arbitrary number) (6.2)

Take a difference of both sides of the equations;

a (x) **y** = **b** – λ

Then

y = 1/**a** (x) (**b** - λ)

= - **a**/a^2 (x) ((**b** - λ)

= **b** x **a** /a^2 + (**a.b**)/a^2 +(λ /a^2).**a**

In order for **y** to be a pure vector, it must hold

(**a.b**) = 0

The solution is;

y = **b** x **a** /a^2 + (λ /a^2).**a** (6.3)

Note **y** is also the solution of equation (6.2). In that case λ is given and **b** is an arbitrary vector.

[3] Linear equation of Vector-Scalar with right and left side multiplications

$$[a] (x) [x] + [x] (x) [b] = [c]$$

Decomposed to components as follows

$$(a0+\mathbf{a}) (x) (x0 +\mathbf{x}) + (x0+\mathbf{x}) (x) (b0+ \mathbf{b}) = c0 + \mathbf{c}$$

Furthermore

Scalar part;

$$(a0+b0).x0 - ((\mathbf{a}+\mathbf{b}).\mathbf{x}) = c0$$

Vector part;

$$(a0+b0).\mathbf{x} + x0.(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\times \mathbf{x} = \mathbf{c}$$

The second term is in the inner plane spanned by vector **a**,**b** and the third tem gives a vector on the transversal plane orthogonal the inner plane. We discriminate vectors in the inner and transversal planes by suffixes i and t as follows;

$$(a0+b0).x0 - ((\mathbf{a}+\mathbf{b}).\mathbf{x}i) = c0$$

$$(a0+b0).\mathbf{x}i + x0.(\mathbf{a}+\mathbf{b}) = \mathbf{c}i$$

$$(a0+b0).\mathbf{x}t + (\mathbf{a}-\mathbf{b})\times\mathbf{x}i = \mathbf{c}t$$

where

$$\mathbf{c}t = (a+b)/|a+b| \text{ x } \{ \mathbf{c} \text{ x } (a+b)/|a+b| \}$$

$$= (a-b)/|a-b| \text{ x } \{ \mathbf{c} \text{ x } (a-b)/|a-b| \}$$

$$\mathbf{c}i = \mathbf{c} - \mathbf{c}t$$

The above equations are re-written as follows;

Let

$$\alpha i = ai + bi \quad (i = 0,1,2,3)$$

$$\beta i = ai - bi \quad (i = 0,1,2,3)$$

then

$$\alpha 0 \cdot x0 - (\boldsymbol{\alpha} \cdot \mathbf{x}i) = c0$$

$$\alpha 0 \cdot \mathbf{x}i + x0 \cdot \boldsymbol{\alpha} = \mathbf{c}i$$

$$\alpha 0 \cdot \mathbf{x}t + \boldsymbol{\beta} \times\mathbf{x}i = \mathbf{c}t \qquad =$$

The solutions are;

$$\mathbf{x}= \mathbf{x}i + \mathbf{x}t$$

- If $\alpha 0$ =/= 0,

$$x0 = (\alpha 0 \cdot c0 + \boldsymbol{\alpha} \cdot \mathbf{c}i) / |[\alpha]|^2$$

$$\mathbf{x}i = (\mathbf{c}i - x0 \cdot \boldsymbol{\alpha}) / \alpha 0$$

$$\mathbf{x}t = (\mathbf{c}t - \boldsymbol{\beta} \times \mathbf{x}i) / \alpha 0$$

where

$$\boldsymbol{\alpha} = (\alpha 1, \alpha 2, \alpha 3)$$

$$\boldsymbol{\beta} = (\beta 1, \beta 2, \beta 3)$$

$$|[\alpha]|^2 = \alpha 0^2 + \alpha 1^2 + \alpha 2^2 + \alpha 3^2$$

- If $\alpha 0 = 0$

$$\boldsymbol{\alpha} \cdot \mathbf{x}i = - c0$$

$$x0 \cdot \boldsymbol{\alpha} = \mathbf{c}i$$

$$\boldsymbol{\beta} \times\mathbf{x}i = \mathbf{c}t \qquad =$$

If vector $\boldsymbol{\alpha} = \mathbf{a} + \mathbf{b}$ does not share the same direction as vector **c**i, then there is no solution.

If $\boldsymbol{\alpha}$ and **c**i have the same direction, then the solution is given as;

$$x0 = |\mathbf{c}i| / |\boldsymbol{\alpha}|$$

$$\mathbf{x}i = (\mathbf{c}t \times \boldsymbol{\beta} + \lambda \cdot \boldsymbol{\beta}) / (\boldsymbol{\beta} \cdot \boldsymbol{\beta})$$

where

$$\lambda = - \{ (\boldsymbol{\beta} \cdot \boldsymbol{\beta}) \cdot c0 + \boldsymbol{\alpha} \cdot (\mathbf{c}t \times \boldsymbol{\beta})\} / (\boldsymbol{\alpha} \cdot \boldsymbol{\beta})$$

xt can be arbitrary.

6.2. Second degree equations

[1] Basic equation

For unknown [x],

$$[x]^2 + [a] (x) [x] + [x] (x) [a] + [c] = 0$$

Modified to be

$$\{[x] + [a]\}^2 = [a]^2 - [c]$$

Hence

$$[x] = -[a] +- SQR([a]^2 - [c])$$

[2] General cases

$$[x]^2 + [a] (x) [x] + [x] (x) [b] + [c] = 0$$

Let

$$[x] = x0 + \mathbf{x}$$

To modify the equations

Scalar part;

$$x0^2 + (a0+b0).x0 + c0 - \{ (\mathbf{x}.\mathbf{x}) + ((\mathbf{a}+\mathbf{b}).\mathbf{x}) \}= 0$$

Vector part;

$$(2x0+a0+b0).\mathbf{x} + x0.(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\times \mathbf{x} + \mathbf{c} = 0$$

The third term is transversal to the inner plane spanned by vectors **a, b.**

We decompose the equations into inner and transversal planes discriminating terms with suffixes i and t ;

$$(2x0+a0+b0).\mathbf{x}i + x0.(a+b) + (a-b)\times \mathbf{x}t + \mathbf{c}i = 0$$

$$(2x0+a0+b0).\mathbf{x}t + (a-b)\times \mathbf{x}i + \mathbf{c}t = 0$$

Solutions;

If $2x0+a0+b0 = 0$

$$x0 = - (a0+b0)/2$$

$$\mathbf{x}i = \{ \boldsymbol{\beta} \times \mathbf{c}t + \lambda \boldsymbol{\beta} \} / (\boldsymbol{\beta} . \boldsymbol{\beta})$$

$$\mathbf{x}t = \{ \boldsymbol{\beta} \times \mathbf{c}i + x0. \boldsymbol{\beta} \times \boldsymbol{\alpha} \} / (\boldsymbol{\beta} . \boldsymbol{\beta})$$

where

α = a + b
β = a - b
λ ; arbitrary number

If 2x0+ α 0 =/= 0

xi = { λ . β + β × ct - (2x0+ α 0).(x0. α + ci) }
/{(2x0+ α 0)^2 + (β . β)}

xt = { β ×(x0. α + ci)− (2x0+ α 0).ct }
/{(2x0+ α 0)^2 + (β . β)}

where

λ = - {x0.(α . β) + (β .ci)} / (2x0+ α 0)

x0 can be solved from the scalar part, which has higher than 4^{th} degrees, hence cannot be expressed in a closed form.

Recursive solution

In concrete numerical problems it can be solved recursively.

The equation is modified;

[a](x)[x] = − {[x](x)[x] + [x](x)[b] + [c]}

Then the solution is expressed as;

[x] = − 1/[a]
(x) { [x](x)[x] + [x](x)[b] + [c] }

which takes a recursive format.

Let x(n) be the value in the n-th cycle,
Then,

[x(n)] = − 1/[a](x)
{[x(n−1)](x)[x(n−1)] + [x(n−1)](x)[b] + [c]}

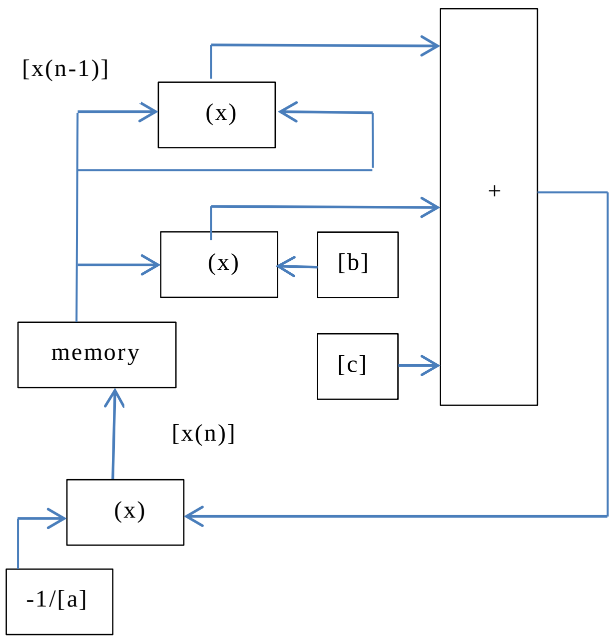
The algorithm is depicted as follows.

The initial value can be any Vector-Scalar [x(0)]. Then [x(1)] is calculated from [x(0)]. The cycle repeats checking the difference [x(n)]-[x(n)].

If [x(n)]-[x(n−1)] turns to 0 for n increasing, then conversion to the solution will be achieved.

Different solutions may be reached from different initial values.

The calculation algorithm is depicted in the following figure.



Recursive solution circuit

Recursive calculation method will be applicable to wide varieties of applications.

References

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