A Brief Review of Numbers and Mathematics

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Sunday, 4, August, 2024

1. Natural Numbers

Natural or whole numbers are 1,2,3,,,,. The natural numbers are for counting things, which originated in primitive ages. It is known that trading was widely made very early as stone tools made of materials from limited places are discovered over far wider areas. The additions (+) and subtractions (-) were probably known in primitive ages.

2. Multiples and Fractions

The advent of agriculture and domestication of animals totally changed the society based on the concept of private ownership. Calculation of the area of a piece of land necessitated multiplication (× or .) and Division (÷ or /). The area S of a rectangle with sides lengths a, b is; S = a × b = a . b. If the land is evenly divided to three parts, each area is S/3. Ancient Egypt needed redistribution of lands after regular flood of River Nile thus practical geometry highly developed. They had the concept of fractions. Note the division is multiplication with fractions; a / b = a . (1/b) .

3. Discovery of zero and negative numbers

The trades of private properties created the concept of personal debt and the corresponding mathematical concept of negative numbers. Discovery of zero was essential to complete the number system with positive and negative numbers.

Zero (0); definition ; for any number a, $a - a = 0$ which means "nothing".

Then, $a = a + 0$, $a = a - 0$

$a . 0 = 0$ $0 / a = 0$ { for $S = a . b$, $0 = S – S = a . b – a . b = a .(b – b) = a . 0$ }

{ $0 / a = 0 . (1/a) = 0$ }

Negative number

Definition $- a = 0 – a$ then $- a + a = 0$

Multiplication $1 \times 1 = 1$

$-1 \times 1 = 1 \times (-1) = -1$

$-1 \times (-1) = 1$

The definition of the multiplication rule is obvious in calculation of the area of a rectangle with sides lengths a - b and c – d;

$$S = (a - b) . (c - d) = a . b + a . (-d) + (-b) . c + (-b) . (-d) = a . b - a . d - b . c + b . d$$

The letters 0,1,2,3,4,5,6,7,8,9 are called Arabic numbers. They enable digital expression of numbers, which is clear, precise and easy to calculate by hand. Just think how to calculate; two hundred and thirty five plus one hundred and twenty three without Arabic numbers. It is not easy, is it? With Arabic numbers; 235 + 123 = 358. So easy! 165 + 78 = 100 + 60 + 5 + 70 + 8 = 100 + 130 + 13 = 243. Other examples are ;

163 -79 = 100 + 60 +3 – (70 +9) = 80 + 70 + 13 -70 -9 = 84.

15 x 27 = (10 + 5) x (20 + 7) = 10x20 + 10x7 + 5x20 + 5x7 = 200 + 70 +100+ 35 = 300 + 105 = 405.

133 / 11 = (110 + 22 +1) /11= 10 +2 + 1/11 = 12 + 1/11. Or 133 = 12x11 +1 ; Quotient is 12, remainder 1.

The Arabic number system spread widely in Middle East and Europe during middle age for its clarity and convenience together with the growth of wide area trading and industry.

4. Integers;

The positive and negative whole numbers and zero form integers. The integers are complete for additions, subtractions and multiplications, but incomplete for divisions. For example, a/b does not exists in integers unless a is divisible by b.

5. Rational numbers

Positive and negative numbers composed of multiples and fractions and zero form rational numbers.

[1] Rational numbers form “field” as an algebraic system

Four operations (addition +, subtraction -, multiplication x, division /) are freely possible with rational numbers and meeting the following laws; .

(1) Associative law ; a + (b + c) = (a + b) + c a . (b . c) = (a . b) . c

(2) Commutative law ; a + b = b + a a . b = b . a

(3) Distributive law ; a . (b + c) = a . b + a . c

[2] Compactness

Between any two rational numbers; a and b, there is always a rational number (a + b) /2 in between. Hence there are infinite rational numbers between any numbers a and b ; [a, b].

Irrational numbers

In ancient Greece it had been believed the rational numbers exhaust all numbers because of the compactness. But irrational umbers were discovered as follows.

It is said Pythagoras discovered the theorem; let a triangle be with sides length a, b, c. If the angle between sides a and b is right angle, then c^2 = a^2 + b^2 (a^2 = a . a). (Pythagoras theorem)
If a = b = 1, then c^2 = 2, or c = √(2), which is not a rational number.

Proof;

Let √(2) = p / q, where p and q are whole numbers mutually prime, i.e. , they have no common divisor except for 1. Squaring both sides we get p^2 = 2 . q^2. Then p must be an even number.
Let p = 2 . r. Then q^2 = 2 . r^2. That is, q must be an even number, too, which contradicts the premise that p and q have no common divisor except for 1.

6. Real numbers

The rational and irrational numbers form real numbers. It is known there are infinitely more irrational numbers than rational numbers. The size of rational numbers is a countable infinity, while that of real numbers is uncountable infinity. Real numbers exhaust all points on a straight line from negative infinity to positive infinity.

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Imaginary numbers

Equations of first degree;

a . x = b ; a , b are known real numbers and x is unknown.

The solution is x = b / a (a =/= 0) exists within real numbers.

On the other hand,

Equations of second degree; a . x^2 + b . x + c = 0 (a=/= 0)

The root; x = { -b +- √(D) } / a ; D = b^2 - 4a.c

In the case D >= 0, Two roots exist within real numbers.

However,

In the case D < 0, there is no solution within real number.

If we define an **imaginary** number i = √(-1) or i^2 = -1,

then x = { -b +- √(-D).i } / a

thus all equations of second degrees is solvable within complex numbers.

7. Complex Numbers

For real numbers a and b, the complex number is defined by c = a + b . i (i^2 = -1)

The complex number is also expressed as c = a + b . i = (a, b)

In the above expression, a is the real part and b is the imaginary part.

Complex conjugate

For complex number $c = a + b.i = (a , b)$, the conjugate number c^* is defined as follows;

$$c^* = a - b.i \quad = (a, -b)$$

Then the real and imaginary part are defined as follows;

Real part; $a = Re (c) = (c + c^*) / 2$

Imaginary part $b = Im (c) = (c - c^*) / (2i)$

Absolute value of complex number

$$c .c^* = (a + bi) . (a - bi) = a^2 + b^2$$

Absolute value of c ; $|c| = \sqrt{(c . c^*)}$

Polar representation

Let z be a complex variable $z = x + i. y = (x , y)$; Cartesian coordinate form

The complex number z can be also expressed in the polar coordinate expression as depicted in the following figures ; .

$z = x + i.y$; Cartesian Coordinate form

$= r. \{ \cos(\theta) + i . \sin(\theta) \}$

; Polar coordinate form

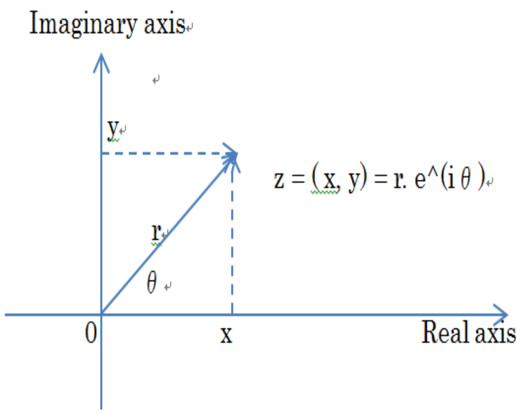
Where

$r^2 = z . z^* = x^2 + y^2$

$r = |z| = \sqrt{(x^2 + y^2)}$

; absolute value or length of z

θ ; argument of z



Euler formula

For the complex variable with the unit length r = 1 ;

$$z = \cos(\theta) + i . \sin(\theta)$$

Differentiate z by θ ,

$$dz/d\theta = - \sin(\theta) + i . \cos(\theta) = i. z$$

The solution is

$$z = \cos(\theta) + i . \sin(\theta) = e^{(i. \theta)}$$

For $\theta = \pi$

$$e^{(i. \pi)} = -1$$; Euler's formula

Operations of Complex Numbers

For complex numbers z and w

$z = x + i.y = (x, y) = r.(\cos(\theta) + i.\sin(\theta)) = r.e\hat{}(i.\theta)$

$w = u + i.v = (u, v) = s.(\cos(\phi) + i.\sin(\phi)) = s.e\hat{}(i.\phi)$

Addition:

$z + w = (x + u, y + v)$

Multiplication;

$z.w = r.s.e\hat{}(i.(\theta + \phi) = (x.u - y.v, x.v + y.u)$

Division ;

$z / w = r / s . e\hat{}(i.(\theta - \phi) = z.w^* / |w|\hat{}2 = (x.u + y.v, -x.v + y.u) / (u\hat{}2 + v\hat{}2)$

Field of complex numbers

Complex numbers form a field as an algebraic system.

That is, the four operations (+, -, x, /) are always possible on complex numbers meeting the associative, commutative and distributive laws.

Completeness of Complex Numbers

Complex numbers are complete as a number system as follows;

Basis Theorem of Algebra (F. Gauss)

Algebraic equations of arbitrary degree; n (natural number) have n solutions within real or complex numbers. Thus any algebraic equation (polynomials) of degree n can be factored as follows.

$P(z) = z\hat{}(n) + c(n-1) . z\hat{}(n-1) +,,,,,, + c(1).z + c(0)$

$= (z - z(n)) . (z - z(n-1)) . \quad . \quad . (z - z(1)) . (z - z(0))$

where z(0), z(1),,,,,z(n) are the roots of the equation P(z) = 0 , which are within complex numbers system. Note the coefficients c(0), c(1),,,,,c(n-1) can be also complex numbers in general.

Historically the complex number took some time to be accepted as numbers even among mathematicians as the term “imaginary” suggests. The complex numbers are today the very basis of mathematics without which no science & technology would be possible.

Conclusion

The numbers systems and mathematics give the basis for science & technology. Their development is closely related with that of the industry and trading in the human society. Its origin is far back in very remote primitive ages. They were invented because they were necessary as the proverb says; necessity is mother of inventions.

Another driving force of the development is the unceasing curiosity and desire of human beings to expand our knowledge to find and solve new problems. New problems are mother of new solutions.